WNE Linear Algebra
Final Exam
Series A

28 January 2020

Please use separate sheets for different problems. Please answer all questions on a single sheet. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks and each question is worth 5 marks. To pass it necessary (but not sufficient) to score at least 50% out of 6 questions.

Problem 1.
Let $V = \text{lin}((1,1,1,1),(1,3,-3,-5),(2,3,0,-1))$ be a subspace of $\mathbb{R}^4$.

a) find a basis $\mathcal{A}$ of the subspace V and the dimension of V,
b) find coordinates of the vector $v = (3,5,-1,-3) \in \mathbb{R}^4$ relative to the basis $\mathcal{A}$.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} 2x_1 + 7x_2 + 3x_3 + 5x_4 = 0 \\ x_1 + 3x_2 + 2x_3 + x_4 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ of the subspace V and the dimension of V,
b) if $v = (2t, -t^2, 2, 1)$ find all $t \in \mathbb{R}$ such that $v \in V$.

Problem 3.
Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear endomorphism given by the formula

$$\varphi((x_1,x_2,x_3)) = (3x_1 - x_2, 2x_1, 3x_1 - 3x_2 + 2x_3).$$

a) find the eigenvalues of φ and bases of the corresponding eigenspaces,
b) find a matrix $C \in M(3 \times 3; \mathbb{R})$ such that

$$C^{-1}M(\varphi)_{st}^{st}C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Problem 4.
Let $\mathcal{A} = ((1,1,2),(0,1,3),(1,0,0))$, $\mathcal{B} = ((1,2),(0,1))$ be ordered bases of $\mathbb{R}^2$. Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformations given by the matrix

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -2 \end{bmatrix},$$

and let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by the formula

$$\psi((x_1,x_2)) = (x_1, x_1 + x_2).$$

a) find the formula of φ,
b) find the matrix $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{B}}$.

Problem 5.
Consider the following linear programming problem $-x_2 - 3x_3 + 2x_5 \to \min$ in the standard form with constraints

$$\begin{cases} x_1 + 2x_2 + 5x_3 + 7x_4 - 5x_5 = 9 \\ x_1 + x_2 + 3x_3 + 4x_4 - 3x_5 = 5 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 5$$

a) which of the sets $\mathcal{B}_1 = \{1, 2\}$, $\mathcal{B}_2 = \{3, 4\}$, $\mathcal{B}_3 = \{3, 5\}$ are basic? Which basic set is basic feasible? Write the corresponding feasible solution.
b) solve the linear programming problem using simplex method.

Questions

Question 1.
Is the matrix $A = \begin{bmatrix} a^2 & a \\ a & 2 \end{bmatrix}$ positive definite for all $a \in \mathbb{R}$?

Question 2.
If $v, w \in, \mathbb{R}^3$, $v \neq \mathbf{0}$, is the image of vector $w \in \mathbb{R}^3$ under the (linear) orthogonal symmetry about the line $V = \operatorname{lin}(v)$ equal to

$$S_V(w) = 2\frac{w \cdot v}{v \cdot v}v - w?$$

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ and $A + A^\mathsf{T} = \mathbf{0}$, does it follow that $A^3 + (A^\mathsf{T})^3 = \mathbf{0}$?

Question 4.
Is it possible that $A, B \in M(2 \times 2; \mathbb{R})$, $\det A \neq 0$, $\det B \neq 0$ but $\det(A + B) = 0$?

Question 5.
Does there exist matrix $A \in (100 \times 3; \mathbb{R})$ with pairwise different rows such that the dimension of the set of all solutions of the equation $Ax = \mathbf{0}$ is equal to 1?

Question 6.
Are the affine subspaces E, $H \subset \mathbb{R}^3$ given by

$$E\colon x_1 + 2x_2 - 3x_3 = 2,$$
$$H = (1, 2, 1) + \operatorname{lin}((1, 2, -3)),$$

perpendicular?